Optimum Selection of Design Variable Increments to Improve Flutter Characteristics

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FOREWORD

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Introduction

In an earlier paper[1], the expression for the partial derivative of an eigenvalue α_m with respect to a parameter p_k was given as

$$\alpha_{m,k} = -\theta_m \left[\alpha_m A_{,k} + B_{,k} \right] \varphi_m \,, \tag{1}$$

for the eigen-problem

$$\alpha_m A \varphi_m + B \varphi_m = 0 \,, \tag{2}$$

where A and B need not be Hermitian, but distinct eigenvalues are required.

When the flutter analysis is formulated as an eigen-problem, the expression for the derivative is applicable. Both modal and collocation flutter analysis methods are accommodated. However, in the case of modal analysis, it happens that some of the required quantities are not directly available and must be developed. The development is relatively straightforward, and is based on the concept of completeness in an engineering sense. With the partials of the eigenvalue, the gradient of the flutter stability constraint surface in parameter space may be developed, thereby making possible automated optimized design which accommodates the flutter constraint.

Modal Analysis Method

In order to provide easy reference and to establish notation, some elementary results will be given. The system is represented by the matrix equation

$$M \ddot{x} + K x = f \,, \tag{3}$$

and the assumed homogeneous solution

$$x = \psi e^{i\Omega t} \tag{4}$$

leads to the eigen-problem

$$-\lambda M\psi + K\psi = 0. \tag{5}$$

Making the coordinate transformation

$$x = \Psi q \tag{6}$$

in equation (3) and premultiplying gives

$$\left[\Psi^T M \Psi\right]\ddot{q} + \left[\Psi^T K \Psi\right] q = \Psi^T f = Q , \tag{7}$$

where Ψ is a matrix whose columns are the eigenvectors ψ. It is noted that the coordinate transformation (6) assumes what is called here completeness in an engineering sense. Under restriction of sinusoidal motion

$$q = \varphi \sin \omega t , \tag{8}$$

the generalized forces are written

$$\{Q\} = \omega^2 [P]\{q\} , \tag{9}$$

where the matrix P is a function of reduced frequency as well as other aerodynamic variables.

The usual procedure is to write the problem

$$\left(\frac{1+ig}{\omega^2}\right)\left[\Psi^T K \Psi\right]\varphi - \left[\left[\Psi^T M \Psi\right] + P\right]\varphi = 0 , \tag{10}$$

where the g represents the amount of structural damping required to maintain sinusoidal motion. Equation (10) is in the form of (2). In practice the aerodynamic coefficients are evaluated at several reduced frequencies, the

corresponding eigenvalues found, and a plot made of the damping required versus speed. The critical flutter speed is found by the required damping being zero. It is assumed that this has been done, and that the aerodynamic matrix P is evaluated at the critical speed or reduced frequency. Obviously, the partial of the eigenvalue of (10) is

$$\alpha_{m,k} = -\frac{2}{\omega_m^3}\,\omega_{m,k} + i\left(\frac{1}{\omega_m^2}\,g_{m,k} - \frac{2}{\omega_m^3}\,g_m\,\omega_{m,k}\right). \tag{11}$$

Proceeding with the development leads to

$$A_{,k} = \Psi_{,k}^T K \Psi + \Psi^T K_{,k} \Psi + \Psi^T K \Psi_{,k} \quad , \tag{12}$$

and

$$-B_{,k} = \Psi_{,k}^T M \Psi + \Psi^T M_{,k} \Psi + \Psi^T M \Psi_{,k} + P_{,k} \; . \tag{13}$$

The expression for the derivative of an eigenvector is given by Fox and Kapoor[2] as

$$\psi_{j,k} = \sum_{\ell} D_{kj\ell}\,\psi_\ell \quad , \tag{14}$$

where

$$D_{km\ell} = \psi_\ell^T\left[K_{,k} - \lambda_m M_{,k}\right]\psi_m \; , \; \ell \neq m, \tag{15}$$

and

$$D_{kmm} = -\frac{1}{2}\,\psi_m^T M_{,k}\,\psi_m \; , \tag{16}$$

provided the eigenvectors have been normalized such that

$$\psi_m^T M \psi_m = 1. \tag{17}$$

Use of equation (14) assumes completeness in an engineering sense in the same way as equation (6). The m^th element of ψ_j is Ψ_{mj}. Therefore, it follows that

$$\Psi_{mj,k} = \sum_{\ell} D_{kj\ell}\,\Psi_{m\ell} \; , \tag{18}$$

or in matrix form

$$\Psi_{,k} = \Psi D_k^T . \tag{19}$$

Now consider an element P_{mj} of the aerodynamic matrix P contained in the expression

$$\omega_m^2 P_{mj} = \iint \Psi_m \Delta p_j \, dA , \tag{20}$$

and take the partial derivative

$$2\omega_m \omega_{m,k} P_{mj} + \omega_m^2 P_{mj,k} = \iint (\Psi_{m,k} \Delta p_j + \Psi_m \Delta p_{j,k}) \, dA . \tag{21}$$

The pressure difference Δp_j caused by unit sinusoidal amplitude in the generalized coordinate corresponding to the eigenvector Ψ_j is found by solving the equation relating it to the down wash magnitude

$$w_j = V_\infty \Psi_j' + i \omega_m \Psi_j = \iint K \Delta p_j \, dA , \tag{22}$$

where K is the known kernel function. Implicit in the development here is the usual assumption that the pressure due to a general downwash may be written as a linear combination of Δp_j. Taking the partial of (22) gives

$$w_{j,k} = V_\infty \Psi_{j,k}' + i \omega_{m,k} \Psi_j + i \omega_m \Psi_{j,k} = \iint (K_{,\omega_m} \omega_{m,k} \Delta p_j + K \Delta p_{j,k}) dA , \tag{23}$$

or, by using (14) and (22)

$$w_{j,k} = \sum_\ell D_{kj\ell} w_\ell + i \omega_{m,k} \Psi_j . \tag{24}$$

Assuming an expression in the convenient form

$$\Delta p_{j,k} = \sum_\ell (D_{kj\ell} + \omega_{m,k} E_{kj\ell}) \Delta p_\ell \tag{25}$$

(justified again by assumed engineering completeness) and writing

$$z_j := \iint K_{,\omega_m} \Delta p_j \, dA , \tag{26}$$

then substituting into (23) yields

$$w_{j,k} = \omega_{m,k} z_j + \iint K \sum_\ell (D_{kj\ell} + \omega_{m,k} E_{kj\ell}) \Delta p_\ell \, dA , \tag{27}$$

which, by using (22), reduces to

$$w_{j,k} = \omega_{m,k} z_j + \sum_{\ell} D_{kj\ell} w_{\ell} + \omega_{m,k} \sum_{\ell} E_{kj\ell} w_{\ell} . \tag{28}$$

Equating (24) and (28) and simplifying leads to

$$i \psi_j - z_j = \sum_{\ell} E_{kj\ell} w_{\ell} \quad ; \omega_{m,k} \neq 0 , \tag{29}$$

from which the $E_{kj\ell}$ may be found by equating at specific points, and solving in matrix form. Expression (21) becomes

$$2\omega_m \omega_{m,k} P_{mj} + \omega_m^2 P_{mj,k} = \sum_{\ell} D_{km\ell} \omega_m^2 P_{\ell j} + \sum_{\ell} \left(D_{kj\ell} + \omega_{m,k} E_{kj\ell} \right) P_{m\ell} ,$$

from which, in matrix form,

$$P_{,k} = D_k P + P D_k^T + \omega_{m,k} P \left[E_k^T - \frac{2}{\omega_m} I \right] . \tag{30}$$

Collecting the above results, equation (1) becomes

$$\begin{aligned} \alpha_{m,k} = &- \theta_m^T \Big[\frac{1}{\omega_m^2} \left[D_k [\Psi^T K \Psi] + \Psi^T K_{,k} \Psi + [\Psi^T K \Psi] D_k^T \right] \\ &- \left[D_k + \Psi^T M_{,k} \Psi + D_k^T + D_k P + P D_k^T \right] \Big] \varphi_m \\ &- \omega_{m,k} \theta_m^T P \left[E_k^T - \frac{2}{\omega_m} I \right] \varphi_m , \end{aligned} \tag{31}$$

where

$$\theta_m^T = \varphi_m^T \left[[\Psi^T K \Psi] \varphi_m \varphi_m^T \right]^{-1} . \tag{32}$$

Numerical evaluation of (31) may be expressed as

$$\alpha_{m,k} = a_R + i\, a_I + \omega_{m,k} \left(b_R + i\, b_I \right) . \tag{33}$$

Equating real parts of (11) and (33), and solving gives

$$\omega_{m,k} = - \frac{a_R}{b_R + \frac{2}{\omega_m^3}} , \tag{34}$$

and similarily, the imaginary parts (with $g_m = 0$) lead to

$$g_{m,k} = \omega_m^2 \left(a_I + \omega_{m,k} b_I \right) . \tag{35}$$

Collocation Analysis Method

The eigen-problem for the collocation flutter analysis method is written[3]

$$\frac{1+ig}{\omega^2} h = C[M+R]h, \tag{36}$$

where C is the structural influence coefficient matrix, and R is a complex matrix of aerodynamic influence coefficients, which are functions of reduced frequency. Again, it is assumed that the aerodynamic matrix is evaluated at the critical condition.

Proceeding directly to evaluation of equation (1), the derivative of the eigenvalue is

$$\beta_{m,k} = e_m^T \left[C_{,k} [M+R] + C M_{,k} \right] h_m , \tag{37}$$

where

$$e_m^T = h_m^T \left[h_m h_m^T \right]^{-1} . \tag{38}$$

It may be easier to find e_m^T directly from the eigen-problem

$$\frac{1+ig}{\omega^2} e = [M+R^T]C \; e, \tag{39}$$

rather than to perform the inversion of the large matrix in (38). Expressions analogous to (33), (34), and (35) may be developed.

Implementation of Frequency Ratio Limit

It is commonly known that flutter speed is sensitive to the ratio of the two structural natural frequencies which couple to produce flutter. Johnson and Warren[4] have suggested that a limit be placed on the ratio to approximate the flutter constraint. Flutter analyses during preliminary design establish the frequency ratio limit, ρ_c. When the eigen-problem of

the structure using the structural influence coefficient is written

$$-\lambda CM\Psi + \Psi = 0 , \tag{40}$$

the frequency ratio limit approximation to the flutter constraint may be expressed as

$$\frac{\lambda_i}{\lambda_j} = \rho \leq \rho_c . \tag{41}$$

It follows that

$$\rho_{,k} = \frac{\lambda_j \lambda_{i,k} - \lambda_i \lambda_{j,k}}{\lambda_j^2} , \tag{42}$$

where

$$\lambda_{\ell,k} = \left(\Psi_\ell^T [I - \lambda_\ell CM] \Psi_{\ell,k} - \lambda_\ell \Psi_\ell^T [C_{,k} M + CM_{,k}] \Psi_\ell\right) \lambda_\ell \Psi_\ell^T \Psi_\ell , \tag{43}$$

and where $\Psi_{\ell,k}$ may be computed using the Fox and Kapoor[2] expressions with

$$K = C^{-1} , \quad K_{,k} = - C^{-1} C_{,k} C^{-1} , \tag{44}$$

or perhaps the $\Psi_{\ell,k}$ may be taken equal to zero as suggested by Johnson and Warren. A gradient may be developed from (42) and used in an automated design procedure.

Discussion

From the physical meaning of the symbol g, it is clear that, at the critical airspeed, a qualitative improvement is indicated by the quantity $g_{m,k}$ being negative. This indicates that, in effect, structural damping is added to the system by a positive increment in the design variable under consideration. Comparison of the derivatives of the damping with respect to the various parameters will give some indication of their relative effects. It remains to place matters on a quantitive basis.

The eigenvalue of interest is a function of the reduced frequency and of the design variables, or alternatively, of the airspeed and design variables. Then the increment in the eigenvalue due to increments in the design variables and in airspeed is approximated by

$$\Delta\alpha_m \simeq \sum_k \left(-\frac{2}{\omega_m^3}\,\omega_{m,k} + i\left(\frac{g_{m,k}}{\omega_m^2} - \frac{2 g_m \omega_{m,k}}{\omega_m^3} \right)\right) \Delta p_k$$
$$+ \left(-\frac{2}{\omega_m^3}\,\frac{\partial \omega_m}{\partial V} + i\left(\frac{1}{\omega_m^2}\,\frac{\partial g_m}{\partial V} - \frac{2 g_m}{\omega_m^3}\,\frac{\partial g_m}{\partial V} \right)\right) \Delta V . \tag{45}$$

To maintain the critical condition, the increment in the imaginary part must be zero (also $g_m = 0$)

$$\sum_k g_{m,k}\,\Delta p_k + \frac{\partial g_m}{\partial V}\,\Delta V = 0 , \tag{46}$$

from which

$$\frac{\partial V}{\partial p_k} = -\frac{\partial g_m}{\partial p_k} \Big/ \frac{\partial g_m}{\partial V} , \tag{47}$$

where the variation of damping required as a function of airspeed is obtained from the flutter analyses. Equation (47) may be used to develop the gradient to the constraint and to select a design variable increment size.

In practice, it may be more efficient to use an approximate constraint (guided by previous flutter analyses) such as the frequency ratio of Johnson and Warren[4] to approach the optimum point, then to use the present method to finalize the design.

In the development of equation (1), the assumption of distinct eigenvalues is made. Since there is often a coalescence of frequencies as the critical flutter point is approached, there may be a practical numerical problem caused by close eigenvalues.

Because of the appeal to engineering intuition in the development, there is a need to verify the results of this paper by analyzing typical problems.

Summary

Expressions developed for the derivative of the eigenvalue in the flutter analysis problem offer potential both for automated optimum design and for less sophisticated design iteration.

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